Last time: Path Integral

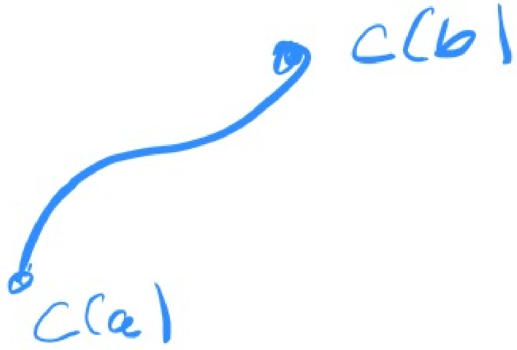
→ integrated a function $f: \mathbb{R}^n \to \mathbb{R}$

along a curve $c: [a,b] \to \mathbb{R}^n$

Today: Line Integral

integrate a vector field $F: \mathbb{R}^n \to \mathbb{R}^n$

along a curve $c: [a,b] \to \mathbb{R}^n$

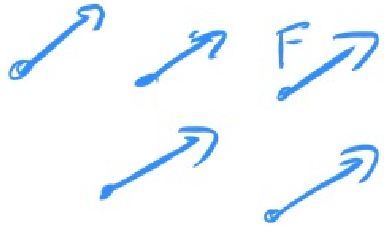


Motivation

would like to calculate work done by a particle moving along curve $c(t)$ in force field given by vector field F

simple case:

- F constant vector field



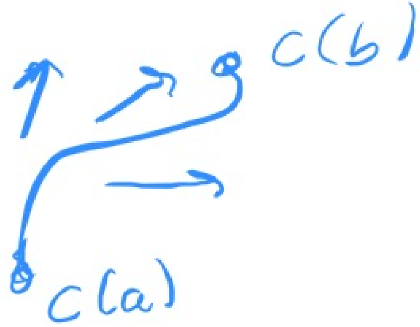
- c a straight line given by vector $\vec{d}$

physics: work $= F \cdot \vec{d}$

complicated case:

F varies

c complicated curve



as for path integral.

replace $\vec{d}$ by $c'(t)$

Get definition of line integral

Def. Let $c: [a,b] \to \mathbb{R}^n$ be a curve

$F: \mathbb{R}^n \to \mathbb{R}^n$ a vector field

Then the line integral is defined by

$$\boxed{\int_c F \cdot ds = \int_a^b F(c(t)) \cdot c'(t)\, dt}$$

Examples

① $n=2$

$$F(x,y) = (x^2, xy)$$

$$c : [0,3] \longrightarrow \mathbb{R}^2$$

$$c(t) = (t, t^2)$$

$$\Rightarrow \int_c F \cdot ds = \int_0^3 F(c(t)) \cdot c'(t)\, dt$$

$$= \int_0^3 F(\underset{x}{\overset{\shortparallel}{t}}, \underset{y}{\overset{\shortparallel}{t^2}}) \cdot (1, 2t)\, dt$$

$$= \int_0^3 (t^2, t^3) \cdot (1, 2t)\, dt$$

$$= \int_0^3 t^2 + 2t^4\, dt = \text{easy}$$

② Different Notation

If F has coordinates F_1, F_2, F_3 $(n=3)$

then the line integral is often also written as

$$\int_C F_1\,dx + F_2\,dy + F_3\,dz$$

e.g. our example can be written as

$$\int_C x^2\,dx + xy\,dy$$

$$\Big(F(x,y) = (\underset{F_1}{\underset{\shortparallel}{x^2}}, \underset{F_2}{\underset{\shortparallel}{xy}}) \Big)$$

(no dz as $n=2$)

Example: $\int_C -y\,dx + x\,dy$

$$\left.\begin{array}{l} c: [0,\pi] \to \mathbb{R}^2 \\ c(t) = (\cos t, \sin t) \end{array}\right\} \quad F(x,y) = (-y, x)$$

$$\int_C \underset{\underset{\sin t}{\shortparallel}}{-y}\,dx + \underset{\underset{\cos t}{\shortparallel}}{x}\,dy = \int_0^{\pi} (-\sin t)(-\sin t) + \cos t \cdot \cos t \, dt$$

$$\frac{dx}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\Rightarrow \boxed{\begin{aligned} dx &= -\sin t \, dt \\ dy &= \cos t \, dt \end{aligned}}$$

$$= \int_0^{\pi} \underbrace{\sin^2 t + \cos^2 t}_{=1} \, dt = \pi$$

Properties of Line Integral

many different ways to parametrize a given physical curve $\mathcal{C}$ via a parametrization $c : [a,b] \to \mathbb{R}^n$ i.e. $\mathcal{C}$ = image of c $= c([a,b])$

Example: $\mathcal{C}$ upper semicircle

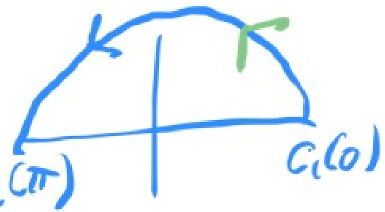
different parametrizations:

$c_1 : [0,\pi] \to \mathbb{R}^2$, $\quad c_1(t) = (\cos t, \sin t)$

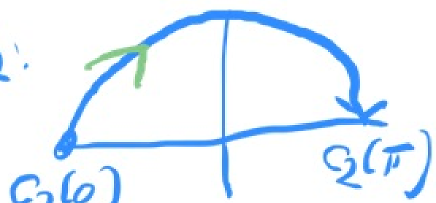
$c_2 : [0,\pi] \to \mathbb{R}^2$, $\quad c_2(t) = (-\cos t, \sin t)$

$c_3 : [0,\pi/2] \to \mathbb{R}^2$, $\quad c_3(t) = (\cos 2t, \sin 2t)$

c_1:



c_2:



c_2: opposite orientation of c_1

c_3

c_3: like c_1 "twice speed" of c_1

Question: How does the line integral depend on choice of orientation?

Theorem (a) The value of the line integral $\int_C F \cdot ds$ does not change if we choose a different parametrization IF it has the same orientation

(b) If c_1 and c_2 parametrize C with opposite orientations

$$\Rightarrow \int_{c_1} F \cdot ds = - \int_{c_2} F \cdot ds$$

Check for our example of semicircle with $F(x,y) = (-y, x)$

$F(x,y) = (-y, x)$

$c_1(t) = (\cos t, \sin t)$

$c_2(t) = (-\cos t, \sin t)$

have seen opposite orientations

$$\int_{c_1} F \cdot ds = \int_0^{\pi} F(\underbrace{\cos t}_{x}, \underbrace{\sin t}_{y}) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{\pi} 1\, dt = \boxed{\pi}$$

$$\int_{c_2} F \cdot ds = \int_0^{\pi} \underbrace{F(-\cos t, \sin t)}_{F(c_2(t))} \cdot \underbrace{(\sin t, \cos t)}_{c_2'(t)}\, dt$$

$$= \int_0^{\pi} (-\sin t, -\cos t) \cdot (\sin t, \cos t)\, dt$$

$$= \int_0^{\pi} (-1)\, dt = \boxed{-\pi}$$